Heavy-quark condensate at zero and finite temperatures for various forms of the short-distance potential

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With the use of the world-line formalism, the heavy-quark condensate in the SU(N)-QCD is evaluated for the cases when the next-to-$1/r$ term in the quark-antiquark potential at short distances is either quadratic, or linear. In the former case, which takes place in the stochastic vacuum model, the standard QCD-sum-rules result is reproduced. In the other case, the condensate turns out to be UV-finite only in less than four dimensions. This fact excludes a possibility to have, in four dimensions, the linear term in the potential, as well as short strings, at the distances smaller than the vacuum correlation length. The use of the world-line formalism enables one to generalize further both results for the condensate to the case of finite temperatures. A generalization of the QCD-sum-rules result to the case of an arbitrary number of space-time dimensions is also obtained and turns out to be UV-finite, provided this number is smaller than six.

1. INTRODUCTION

In this talk, we will briefly review the study performed in Ref. [1]. Its goal was to evaluate, at zero and finite temperatures, the heavy-quark condensate in various confining theories, where the form of the short-distance quark-antiquark potential was different. This can be done by virtue of the formula

$$\left\langle\bar\psi\psi\right\rangle=-\frac{1}{V}\frac{\partial}{\partial m}\left\langle\Gamma\left[A_\mu^a\right]\right\rangle,$$

where m is the current quark mass, V is the four-volume occupied by the system, $\langle\ldots\rangle$ is defined with respect to the Euclidean Yang-Mills action, and the one-loop quark self-energy, i.e. the averaged one-loop effective action of a spin-$\frac{1}{2}$ quark reads [2] (see [3] for a recent review):

$$\left\langle\Gamma\left[A_\mu^a\right]\right\rangle=-2\int\limits_{\Lambda^{-2}}^{\infty}\frac{dT}{T}\mathrm{e}^{-m^2T}\int\limits_P\mathcal{D}x_\mu\int\limits_A\mathcal{D}\psi_\mu\times$$

$$\times\exp\left[-\int\limits_0^T d\tau\left(\frac14\dot x_\mu^2+\frac12\psi_\mu\dot\psi_\mu\right)\right]\left\{\left\langle\mathrm{tr}\,\mathcal{P}\exp\left[ig\int\limits_0^T d\tau\left(A_\mu\dot x_\mu-\psi_\mu\psi_\nu F_{\mu\nu}\right)\right]\right\rangle-N\right\}.\qquad(1)$$

Here, Λ stands for an UV momentum cutoff, the subscripts P and A imply the periodic and antiperiodic boundary conditions respectively, ψ_μ's are antiperiodic Grassmann functions (superpartners of x_μ's), and $A_\mu\equiv A_\mu^aT^a$ with T^a's standing for the generators of the SU(N)-group in the fundamental representation.

To calculate the path integral (1), we will transform it to the one in an effective *constant* Abelian field, to be averaged over. The weight of the average is prescribed by the form of the heavy-quark fundamental Wilson loop, $\langle W(C)\rangle\ \equiv\ \left\langle\mathrm{tr}\,\mathcal{P}\exp\left(ig\int\limits_0^T d\tau A_\mu\dot x_\mu\right)\right\rangle$, in the *original* confining theory. For heavy (namely, c, b, and t) quarks, $\sqrt{S_{\min}(C)}$ is smaller than the vacuum correlation length, where $S_{\min}(C)$ is the area

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of the minimal surface bounded by the contour C. For such small-sized Wilson loops, one can write $S^2_{\min} \simeq \frac{1}{2}\Sigma^2_{\mu\nu}$. Here, $\Sigma_{\mu\nu} = \oint\limits_C x_\mu dx_\nu$ is the so-called tensor area, and "$\simeq$" means "for nearly flat contours", that is a reasonable approximation for a heavy quark, as will be justified below. Then, in various confining theories, the following formulae for a small-sized Wilson loop exist:

• Stochastic vacuum model (SVM) [4], [5]:

$$\langle W(C)\rangle \simeq N \exp\left(-\gamma\Sigma^2_{\mu\nu}\right), \tag{2}$$

where $\gamma \equiv \frac{g^2}{8N(d^2-d)}\left\langle F^2\right\rangle$.

• Theories with confinement of the Abelian type:

$$\langle W(C)\rangle \simeq N \exp\left(-\sigma\sqrt{\frac{1}{2}\Sigma^2_{\mu\nu}}\right) \tag{3}$$

with the following values of σ:

– in the SU(N)-version of the weakly coupled 3d Georgi-Glashow model [6] $\sigma = \frac{\pi}{2}\frac{N-1}{\sqrt{N}}g\sqrt{\zeta}$. Here, $\zeta \sim \frac{M_W^{7/2}}{g}\mathrm{e}^{-4\pi\varepsilon M_W/g^2}$ is the monopole fugacity, $1 \le \varepsilon < 2$, g is the electric coupling constant, and M_W is the W-boson mass, $g^2 \ll M_W$;

– in the London limit of the 4d SU(N)-inspired dual Abelian-Higgs–type theory [7] $\sigma = 2\pi(N-1)\eta^2\ln\kappa$, where η is the vacuum expectation value of the dual Higgs field, and κ is the Landau-Ginzburg parameter, $\ln\kappa \gg 1$.

Therefore, under the assumption that the Feynman-Kac formula, which relates the potential of a heavy quark-antiquark pair to the Wilson loop, can be extrapolated up to the distances of the order of the vacuum correlation length, the next-to-$1/r$ term of the potential is $\propto r^2$ in the SVM, whereas it is $\propto r$ in the Abelian-type theories with confinement.

2. HEAVY-QUARK CONDENSATE AT ZERO TEMPERATURE

In the SVM at arbitrary $d \ge 2$, one has by virtue of eqs. (1), (2):

$$\left\langle\Gamma\left[A^a_\mu\right]\right\rangle = -\frac{2NV}{(8\pi\gamma)^{n/2}}\int\limits_{\Lambda^{-2}}^{\infty}\frac{dT}{T}\mathrm{e}^{-m^2T}\times$$

$$\times\left(\prod_{\mu<\nu}\int\limits_{-\infty}^{+\infty}dB_{\mu\nu}\mathrm{e}^{-\frac{B^2_{\mu\nu}}{8\gamma}}\right)\left\{\int\limits_P\mathcal{D}x_\mu\int\limits_A\mathcal{D}\psi_\mu\times\right.$$

$$\times\exp\left[-\int\limits_0^T d\tau\left(\frac{1}{4}\dot{x}^2_\mu+\frac{1}{2}\psi_\mu\dot{\psi}_\mu+\frac{i}{2}B_{\mu\nu}x_\mu\dot{x}_\nu-\right.\right.$$

$$\left.\left.\left.-\,iB_{\mu\nu}\psi_\mu\psi_\nu\right)\right]-\frac{1}{(4\pi T)^{d/2}}\right\}, \tag{4}$$

where $n = \frac{d^2-d}{2}$ is one half of the number of off-diagonal components of the space-time independent field $B_{\mu\nu}$. The expression in the curly brackets can further be recognized as the Euler-Heisenberg-Schwinger Lagrangian, whose small-T (large-m) expansion yields $\{\ldots\} = \frac{1}{(4\pi T)^{d/2}}\left[\frac{T^2}{3}\sum\limits_{\alpha<\beta}B^2_{\alpha\beta}+\mathcal{O}\left(T^4\left(B^2_{\mu\nu}\right)^2\right)\right]$. The actual parameter of this expansion is $\frac{g^2\left\langle F^2\right\rangle}{N(d^2-d)m^4}$ (it is $\mathcal{O}(N^0)$), and the neglected term is $\sim\left(\frac{g^2\left\langle F^2\right\rangle}{N(d^2-d)m^4}\right)^2$. For $d \sim 4$, the expansion holds for c, b, and t quarks, but not for u, d, and s quarks.

One can further see that $\left\langle\bar\psi\psi\right\rangle$ diverges as $\ln\frac{\Lambda}{m}$ for $d=6$ and as $(\Lambda/m)^{d-6}$ for $d>6$, while for $d<6$ it is finite and reads

$$\left\langle\bar\psi\psi\right\rangle = -\frac{m^{d-5}\Gamma\left(3-\frac{d}{2}\right)}{3(4\pi)^{\frac{d}{2}-1}}\alpha_s\left\langle F^2\right\rangle.$$

In particular, $\left\langle\bar\psi\psi\right\rangle\Big|_{d=4} = -\frac{\alpha_s\left\langle F^2\right\rangle}{12\pi m}$, that coincides with the result of Ref. [8], justifying the approximation $S^2_{\min}\simeq\frac{1}{2}\Sigma^2_{\mu\nu}$ for a heavy quark. At $d=2$ and $d=3$, the results read

$$\left\langle\bar\psi\psi\right\rangle\Big|_{d=3} = \frac{m}{4}\left\langle\bar\psi\psi\right\rangle\Big|_{d=2} = -\frac{\alpha_s\left\langle F^2\right\rangle}{12m^2}.$$

In the theories with the Abelian-type confinement, at arbitrary $d \ge 2$, we have

$$\left\langle \Gamma\left[A_\mu^a\right]\right\rangle = -\frac{2NV\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}\sigma^n}\int\limits_{\Lambda^{-2}}^{\infty}\frac{dT}{T}\mathrm{e}^{-m^2T}\times$$

$$\times\left(\prod_{\mu<\nu}\int\limits_{-\infty}^{+\infty}dB_{\mu\nu}\right)\frac{1}{\left(1+\frac{1}{2\sigma^2}B_{\mu\nu}^2\right)^{\frac{n+1}{2}}}\{\ldots\},$$

that is similar to Eq. (4), but with a different measure of average over $B_{\mu\nu}$. The expansion of $\{\ldots\}$ now goes in powers of σ/m^2. For σ of the order of $(440\,\mathrm{MeV})^2$, the expansion again holds for c, b, and t quarks. One can further see that $\left\langle\bar\psi\psi\right\rangle$ diverges as $(\Lambda/m)^{d-4}$ for $d>4$, while $\left\langle\bar\psi\psi\right\rangle\big|_{d=4} = -\frac{5Nm\sigma}{(4\pi)^2}\ln\frac{\Lambda}{m}$. This divergency contradicts the apparent finiteness of the (non-perturbative part of the) heavy-quark condensate and necessitates to attribute some physical meaning to Λ. In the London limit of the 4d SU(N)-inspired dual Abelian-Higgs–type theory, Λ is of the order of the mass of the dual Higgs boson. In 4d QCD, we come to the conclusion that $\langle W(C)\rangle$ may not have the form $\mathrm{e}^{-\sigma S_{\min}(C)}$ up to arbitrarily short distances. There, Λ is of the order of the inverse thickness of a "short string". In terms of the heavy-quark potential, this result means that the linear next-to-$1/r$ term may exist only up to distances not smaller than the vacuum correlation length. Very short strings, of a length much smaller than the vacuum correlation length, are therefore ruled out in 4d QCD, as well as the linear term in the potential at such distances.

Instead, at $d<4$, $\left\langle\bar\psi\psi\right\rangle$ is finite:

$$\left\langle\bar\psi\psi\right\rangle = -\frac{N\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(2-\frac{d}{2}\right)\sigma m^{d-3}}{3\cdot 2^{d-3}\Gamma\left(\frac{n}{2}\right)\pi^{\frac{d+1}{2}}}.$$

In particular, $\left\langle\bar\psi\psi\right\rangle\big|_{d=3} = m\left\langle\bar\psi\psi\right\rangle\big|_{d=2} = -\frac{2N\sigma}{3\pi^2}$.

3. FINITE-TEMPERATURE GENERALIZATIONS

In this section, we will consider the finite-temperature generalizations of the above-obtained results, starting with the SVM at $d=4$. Antiperiodic boundary conditions for quarks can be taken into account upon the multiplication of the zero-temperature heat kernel by the factor $\left[1+2\sum\limits_{n=1}^{\infty}(-1)^n\mathrm{e}^{-\frac{\beta^2n^2}{4T}}\right]$, where $\beta = 1/(\text{temperature})$ [9]. One has

$$\left\langle\bar\psi\psi\right\rangle = \left\langle\bar\psi\psi\right\rangle_0\left[1+2\beta m\sum_{n=1}^{\infty}(-1)^n nK_1(\beta mn)\right],$$

where $\left\langle\bar\psi\psi\right\rangle_0 = -\frac{\alpha_s\left\langle F^2\right\rangle}{12\pi m}$, and $\left\langle F^2\right\rangle$ experiences a drop by a factor of the order of 2 at the deconfinement temperature, $T_c \sim 150\,\mathrm{MeV}$, due to the evaporation of the chromoelectric condensate [10] [2]. For c, b, and t quarks, $T_c \ll m$, and only the first Matsubara mode yields a sufficient contribution: $\left\langle\bar\psi\psi\right\rangle \simeq \left\langle\bar\psi\psi\right\rangle_0\left(1-\sqrt{2\pi\beta m}\mathrm{e}^{-\beta m}\right)$. The correction to the zero-temperature result is therefore exponentially small. Next, when T starts exceeding the temperature of dimensional reduction, $T_{\mathrm{d.r.}}$, the theory becomes three-dimensional. Since, $T_{\mathrm{d.r.}}$ is not larger than $2T_c$ (see e.g. [5]), for c, b, and t quarks a rather broad range of temperatures above $T_{\mathrm{d.r.}}$ exists, where $T \ll m$. At such temperatures, one gets $\left\langle\bar\psi\psi\right\rangle \simeq -\frac{\alpha_s\left\langle F^2\right\rangle T}{12m^2}\left(1-2\beta m\mathrm{e}^{-\beta m}\right)$.

In the Abelian-type theories with confinement, one should first of all note that the radius of a "short string", $r_\perp$, grows with the temperature. Therefore, one may substitute Λ by $r_\perp^{-1}$ only as long as $mr_\perp(T) \ll 1$. For the sake of generality, let us assume that this is true up to temperatures larger than $T_{\mathrm{d.r.}}$, and that $T_{\mathrm{d.r.}} > T_c$ as in QCD. Implying everywhere below the above-mentioned substitution $\Lambda \to r_\perp^{-1}$, we have at $T<T_c$,

$$\left\langle\bar\psi\psi\right\rangle \simeq \left\langle\bar\psi\psi\right\rangle_0\left(1-\sqrt{\frac{2\pi}{\beta m}}\frac{\mathrm{e}^{-\beta m}}{\ln\frac{\Lambda}{m}}\right), \qquad (5)$$

where $\left\langle\bar\psi\psi\right\rangle_0 = -\frac{5Nm\sigma}{(4\pi)^2}\ln\frac{\Lambda}{m}$. At $T>T_c$, only the spatial string tension, σ_s, does not vanish, and the Wilson loop takes the form $\langle W(C)\rangle \simeq N\exp\left[-\sigma_s\left(\Sigma_{12}^2+\Sigma_{13}^2+\Sigma_{23}^2\right)^{1/2}\right]$. Then, at $T\in$

[2] Henceforth, T will denote the temperature, rather than the proper time.

$[T_c, T_{\rm d.r.}]$, $B_{\mu\nu}$ is a 3×3-matrix, but the theory is still four-dimensional. These two facts together yield the following formula for the condensate:

$$\left\langle\bar\psi\psi\right\rangle \simeq -\frac{2Nm\sigma_s}{3\pi^3}\ln\frac{\Lambda}{m}\cdot\left(1-\sqrt{\frac{2\pi}{\beta m}}\frac{{\rm e}^{-\beta m}}{\ln\frac{\Lambda}{m}}\right).(6)$$

The difference of the overall factor $\frac{2Nm\sigma_s}{3\pi^3}$ from the factor $\frac{5Nm\sigma}{(4\pi)^2}$ of Eq. (5) is due to the fact that the temporal string tension vanishes when one passes from $T < T_c$ to $T > T_c$. Naively, one could have expected at $T = T_c + 0$ the factor $\frac{5Nm\sigma_s}{(4\pi)^2}$, that is not the case. Finally, at $T > T_{\rm d.r.}$, the theory becomes fully three-dimensional. There, $\left\langle\bar\psi\psi\right\rangle \simeq -\frac{2N\sigma_s T}{3\pi^2}\left(1-2{\rm e}^{-\beta m}\right)$, as long as $T \ll m$.

4. SUMMARY

By making use of the world-line formalism, we have evaluated the heavy-quark condensate at zero and finite temperatures. The respective heavy-quark Wilson loop has been considered either within the SVM, or within the theories with Abelian-type confinement. The zero-temperature results are the following:

• SVM, $d = 4$: the result of Ref. [8] is reproduced; $\left\langle\bar\psi\psi\right\rangle$ diverges as $\ln\frac{\Lambda}{m}$ at $d = 6$ and as $(\Lambda/m)^{d-6}$ at $d > 6$, while the finite expression is obtained at $2 \le d < 6$.

• Theories with Abelian-type confinement: the finite expression is obtained at $2 \le d < 4$, while $\left\langle\bar\psi\psi\right\rangle$ diverges as $(\Lambda/m)^{d-4}$ at $d > 4$ and as $\ln\frac{\Lambda}{m}$ at $d = 4$. Possible physical meanings of the UV cutoff at $d = 4$:

– 4d SU(N)-inspired dual Abelian-Higgs–type theory: Λ is of the order of the mass of the dual Higgs boson;

– If, in some approximation, QCD can also be considered as a theory belonging to this class, then Λ is the inverse thickness of a “short string”. Therefore, the “short string” and the associated linear next-to-$1/r$ term in the heavy-quark potential may exist only up to distances not smaller than the vacuum correlation length.

Finally, the finite-temperature generalizations of the above-discussed results have also been presented. In general, antiperiodic boundary conditions for quarks produce corrections, which are exponentially small in the parameter βm (both below the temperature of dimensional reduction and in the broad range of temperatures above it). A nontrivial situation appears, in case of the short-distance linear potential, at temperatures lying between the deconfinement critical temperature and the temperature of dimensional reduction. Although the theory is still four-dimensional in this phase, the Wilson loop is that of a 3d theory (since only the spatial string tension survives the deconfinement phase transition). The result for the condensate in this phase is given by Eq. (6).

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